2. Suppose that f : [a, b] → R is continuous and diﬀerentiable on (a, b). Moreover suppose that f is bounded on (a, b). Prove that f is Lipschitz on [a, b].

[Hint: I suggest using the mean value theorem]

Take M = sup{f’(x) : a < x < b}

By MVT there is a c in (a,b) so f(b) – f(a) = f’(c) (b – a)

f’(c) <\_ M 🡪 f(b) – f(a) <= M (b – a) hence f is lipschitz on (a,b)

3. Let p ∈ (0, 1) and deﬁne f(x) = x^p. Prove that 1. f is Lipschitz on [1, ∞) [Hint: I suggest using the mean value theorem]

f’(x) = p/(x^(1-p))

let a>b>=1 be real numbers

F is obviously continuous on [1, ∞) and differentiable on (1,∞)

Sup{f’(x) : x>=1} = p (x=1)

By MVT, there exists a c in (a, b) so

f(b) – f(a) = f’(c)(b – a)

f(b) – f(a) <= p(b – a) since f’(c) <= p

so f is lipschitz on [a,b] and hence on [1,∞)

2. f is not Lipschitz on [0, 1] [Hint: You don’t have to use the mean value theorem!]

Suppose for contradiction that f is Lipschitz on [0,1]

Then for all a,b in [0,1] there is an M in R ( take WOLOG b>a)

So f(b) – f(a) <= M (b – a) 🡪 (f(b) – f(a))/(b – a) <= M

Take lim a🡪b (f(b) – f(a))/(b – a) = f’(b) <= M

But lim x🡪 0 f’(x) = ∞ but M is finite, a contradiction

3. f is uniformly continuous on [0, ∞) [Hint: I suggest not using the mean value

theorem]

Let E>0

f is continuous on [0,1] and [0,1] is compact, so f is uniformly continuous on [0,1]

so there exists a D1>0 so for all a,b in [0,1] |b – a| < D1 🡪 |f(b) – f(a)| < E

we have shown f is uniformly continuous on [1, ∞)

so there exists a D2>0 for all c,d in [1, ∞) |d – c| < D2 🡪 |f(d) – f(c)| < E

Take D = min{D1, D2}

Now take x in [0,1] and y in [1, ∞) (assume x=/= y since it would be trivial)

There is a z so x<z<y

Suppose |x-y| < D

|x-y| < |x-1| + |y-1| < D1 + D2

|f(x) – f(y)| < |f(x) – f(z)|+ |f(y) – f(z)| < E + E